GRAPHS ON 21 EDGES THAT ARE NOT 2–APEX

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Abstract. We show that the 20 graph Heawood family, obtained by a combination of ∇Y and Y∇ moves on K_7, is precisely the set of graphs of at most 21 edges that are minor minimal for the property not 2–apex. As a corollary, this gives a new proof that the 14 graphs obtained by ∇Y moves on K_7 are the minor minimal intrinsically knotted graphs of 21 or fewer edges. Similarly, we argue that the seven graph Petersen family, obtained from K_6, is the set of graphs of at most 17 edges that are minor minimal for the property not apex.

1. Introduction

A graph is n–**apex** if the deletion of n or fewer vertices results in a planar graph. As this property is closed under taking minors, it follows from Robertson and Seymour's Graph Minor Theorem [RS] that, for each n, the n–apex graphs are characterized by a finite set of forbidden minors. For example, 0–apex is equivalent to planarity, which Wagner [W] showed is characterized by K_5 and $K_{3,3}$. For the property 1–apex, which we simply call **apex**, there are several hundreds of forbidden graphs (see [DD], which refers to work of a team led by Kezdy). Since there are likely even more forbidden minors for the 2–apex property, we divide the problem into more manageable pieces by graph size. In an earlier paper [Ma], the second author showed that every graph on 20 or fewer edges is 2–apex. This means there are no forbidden minors with 20 or fewer edges. In the current paper, we show that there are exactly 20 obstruction graphs for 2–apex of size at most 21.

Following [HNTY], the **Heawood family** will denote the set of 20 graphs obtained from K_7 by a sequence of zero or more ∇Y or Y∇ moves. Recall that a ∇Y **move** consists of deleting the edges of a 3-cycle abc of graph G, and adding a new degree three vertex adjacent to the vertices a, b, and c. The reverse, deleting a degree three vertex and making its neighbors adjacent, is a Y∇ **move**. The Heawood family is illustrated schematically in Figure 1 (taken from [GMN]) where K_7 is graph 1 at the top of the figure and the $(14, 21)$ Heawood graph is graph 18 at the bottom.

Our main theorem is that the Heawood family is precisely the obstruction set for the property 2–apex among graphs of size at most 21. We will state this in terms of minor minimality. We say H is a **minor** of graph G if H is obtained by contracting edges in a subgraph of G. The graph G is **minor minimal** with respect to a graph property $\mathcal{P}$, if G has $\mathcal{P}$, but no proper minor of G does. We call obstruction graphs for the 2–apex property **minor minimal not** 2**–apex or MMN2A**.

Theorem 1.1. *The 20 Heawood family graphs are the only MMN2A graphs on 21 or fewer edges.*

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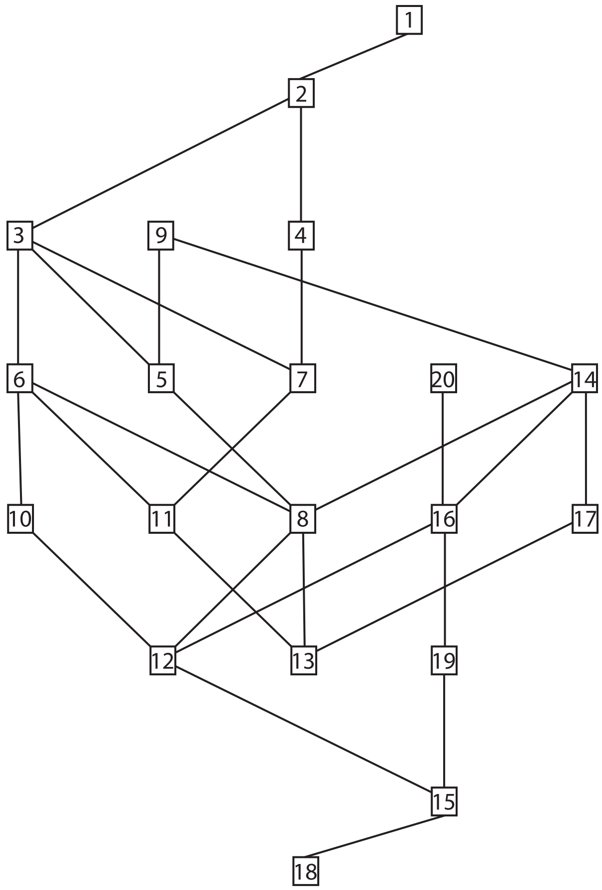

FIGURE 1. The Heawood family (figure taken from [GMN]). Edges represent ∇Y moves.

As there are no MMN2A graphs of size 20 or less [Ma] and one easily verifies that the Heawood family graphs are MMN2A, the argument comes down to showing no other 21 edge graph enjoys this property. We give a more complete outline of our proof at the end of this introduction.

Our interest in 2–apex stems from the close connection with intrinsic knotting. A graph is **intrinsically knotted or IK** if every tame embedding of the graph in $\mathbb{R}^3$ contains a non-trivially knotted cycle. Then, a **minor minimal IK or MMIK** graph is one that is IK, but such that no proper minor has this property. Again, Robertson and Seymour's Graph Minor Theorem [RS] implies a finite list of MMIK graphs, but determining this list or even bounding its size has proved very difficult. Restricting by order, it follows from Conway and Gordon's seminal paper [CG] that K_7 is the only MMIK graph on seven or fewer vertices; two groups [CMOPRW] and [BBFFHL] independently determined the MMIK graphs of order eight; and we have announced (see [Mo] and [GMN]) a classification of nine vertex graphs, based on a computer search. In terms of edges, we know ([JKM] and, independently, [Ma]) that a graph of size 20 or less is not IK. Using the following lemma, (due,

independently, to two research teams) this follows from the lack of MMN2A graphs of that size.

Lemma 1.2. [BBFFHL, OT] *If G is IK, then G is not 2–apex.*

The current authors [BM] and, independently, Lee et al. [LKLO] classified the 21 edge MMIK graphs. These are the 14 KS graphs obtained by ∇Y moves on K_7, first described by Kohara and Suzuki [KS]. In other words, these are the Heawood family graphs except those labelled 9, 14, 16, 17, 19, and 20 in Figure 1. In light of Lemma 1.2, we have a new proof as a corollary to our main theorem.

Corollary 1.3. *The 14 KS graphs are the only MMIK graphs on 21 or fewer edges.*

Proof. Kohara and Suzuki [KS] showed that the KS graphs are MMIK. Suppose G is MMIK of at most 21 edges. Then G is connected. By Lemma 1.2, G has an MMN2A minor and by Theorem 1.1 this means a Heawood family graph minor. As G has at most 21 edges and is connected, G *is* a Heawood family graph. Finally, Goldberg et al. [GMN] and Hanaki, Nikkuni, Taniyama, and Yamazaki [HNTY], independently, showed that in the Heawood family only the KS graphs are IK. Therefore, G is a KS graph. □

The proof of our main theorem relies on our classification of MMNA graphs (i.e., obstructions to the 1–apex, or apex, property) of small order, a result that may be of independent interest. Recall that, in analogy with the Heawood family, the Petersen family is the seven graphs obtained from the Petersen graph by a sequence of ∇Y or Y∇ moves.

Theorem 1.4. *The seven Petersen family graphs are the only MMNA graphs on 16 or fewer edges*

Famously, the Petersen family is precisely the obstruction set to intrinsic linking [RST]. It would be nice to have a similar description of the Heawood family. Theorem 1.1 is one such characterization. As a second corollary to our main theorem, we give a characterization of similar flavor. Hanaki, Nikkuni, Taniyama, and Yamazaki [HNTY] showed that the Heawood family graphs are minor minimal for intrinsically knotted or completely 3-linked or MMI(K or C3L).

Corollary 1.5. *The 20 Heawood family graphs are the only MMI(K or C3L) graphs on 21 or fewer edges.*

Proof. Hanaki et al. [HNTY] proved these graphs are MMI(K or C3L). Let G be MMI(K or C3L) on 21 or fewer edges. Then G is connected. By [HNTY, Remark 4.5], I(K or C3L) implies N2A, so G must have a MMN2A minor. By Theorem 1.1, this means a Heawood minor. It follows that G has 21 edges and is a Heawood family graph, as required. □

This gives two characterizations of the Heawood family. However, like our Theorem 1.4, they are less than ideal due to the hypothesis on graph size. Is there a "natural" description of the Heawood family analogous to the way the Petersen family is precisely the obstruction set for intrinsic linking?

Note that the condition on graph size in these three results is necessary. Indeed, for Theorem 1.4, the disjoint union $K_{3,3} \sqcup K_{3,3}$ is an 18 edge MMNA graph outside the Petersen family. On the other hand, a computer search [P] shows that Theorem 1.4 could be extended to 17 edges: there are no MMNA graphs of size

17. Since IK implies both N2A (Lemma 1.2) and I(K or C3L) (see [HNTY]) there are many examples of MMN2A and MMI(K or C3L) graphs on 22 edges, including $K_{3,3,1,1}$. Foisy [F] showed this graph is MMIK, which means it is also N2A and I(K or C3L). As any proper minor of $K_{3,3,1,1}$ would have at most 21 edges, and no Heawood family graph is a minor, it follows from Theorem 1.1 and Corollary 1.5, that $K_{3,3,1,1}$ is both MMN2A and MMI(K or C3L). So, the hypothesis on size is necessary for both the theorem and its corollary.

Thus, $K_{3,3,1,1}$ and the 14 KS graphs are examples of graphs that enjoy all three properties: MMN2A, MMIK, and MMI(K or C3L). On the other hand, the remaining six Heawood graphs show that a graph can be MMN2A and not MMIK. This includes the graph that we have called E_9 [Ma] and that Hanaki et al. [HNTY] label N_9. In [GMN] we showed that adding an edge to this graph makes it MMIK. In other words, $E_9 + e$ is MMIK and not MMN2A (as it has the N2A graph E_9 as a subgraph). On the other hand, since IK implies I(K or C3L), every MMIK graph has a minor that is MMI(K or C3L) although E_9, for example, shows that the set of I(K or C3L) graphs is a strictly larger class than IK. Similarly, I(K or C3L) implies N2A [HNTY], which means every MMI(K or C3L) has a MMN2A minor, while the disjoint union of three $K_{3,3}$'s is an example of a graph that is N2A but not I(K or C3L).

All six of the Heawood graphs that are not MMIK are MMI(K or C3L) and we can ask if a graph that is MMN2A and not MMIK need be I(K or C3L). However, the disjoint union $G = K_6 \sqcup K_5$ is a counterexample. Since K_6 is MMNA and K_5 is non-planar, G is N2A and, since any proper minor is 2–apex, it is in fact MMN2A. On the other hand, G is neither IK nor I(K or C3L) as each component has fewer than 21 edges.

We conclude this overview of connections between apex graphs and intrinsic knotting with a question. In [GMN] we describe the known 263 examples of MMIK graphs. By Lemma 1.2, none of these graphs are 2–apex. However, it is straightforward to verify that each is 3–apex. Does this hold more generally?

Question 1.6. *Is every MMIK graph 3–apex?*

The remainder of our paper is a proof of Theorem 1.1. Let G be a MMN2A graph of size 21. We must show G is a Heawood family graph. We can assume $\delta(G)$, the **minimum degree**, is at least three. Indeed, in a N2A graph, deleting a degree zero vertex or contracting an edge of a vertex of degree one or two will result in a N2A minor. We can also bound the number of vertices. As G has 21 edges and minimum degree at least three it has at most 14 vertices. On the other hand, we classified MMN2A graphs on nine or fewer vertices in [Ma]. So we can assume $10 \leq |V(G)| \leq 14$. After introducing some preliminary lemmas, and proving Theorem 1.4, in the next section, we devote one section each to the five cases where the number of vertices runs from 14 down to ten. We opted for this reverse ordering as it roughly corresponds to increasing length of the proofs.

2. Preliminaries

We denote the order of a graph G by $|G|$ and its size by $\|G\|$ and frequently use the pair $(|G|, \|G\|)$ as a way of describing the graph. For $a, b \in V(G)$, we will use $G - a$ and $G - a, b$ to denote the induced subgraphs on $V(G) \setminus \{a\}$ and $V(G) \setminus \{a, b\}$, respectively. We will also write $G + a$ to denote a graph with vertices

$V(G) \cup \{a\}$ that includes G as the induced subgraph on $V(G)$. In case $V(G)$ and $\{a\}$ are included in the vertex set of some larger graph, $G+a$ will mean the induced subgraph on $V(G) \cup \{a\}$. We use $N(a)$ to denote the **neighborhood of vertex** a, the set of vertices adjacent to a. We will write NA, MMNA, N2A, and MMN2A for "not apex", (equivalently, "not 1–apex") "minor minimal not apex", "not 2–apex", and "minor minimal not 2–apex" respectively.

Vertices of degree less than three do not participate in determining whether or not a graph is n–apex, so we next describe a systematic way of deleting those vertices.

Definition 2.1. *The* **simplification** G^s *of a graph* G *is the graph obtained by the following procedure.*

(1) *Delete all degree 0 vertices*
(2) *Delete all degree 1 vertices and their edges*
(3) *If there remain vertices of degree 0 or 1, go to step (1)*
(4) *For each degree 2 vertex* v*, delete it and its two edges* va *and* vb*. If* ab *is not already an edge of the graph, add* ab*.*
(5) *If there remain any vertices of degree 0 or 1, go to step (1)*

The procedure allows us to recognize $V(G^s)$ *as a subset of* $V(G)$*. We call these vertices of* G *the* **branch vertices***.*

Note that G^s is a minor of G and is unique, up to isomorphism [P].

Lemma 2.2. *The graph* G *is* n*–apex if and only if* G^s *is.*

Proof. This follows as n–apex is preserved by each step in the definition. □

This means that graphs where G^s is non-planar will be of particular interest. An important class of graphs with $G^s = K_{3,3}$ are the **split** $K_{3,3}$**'s**: graphs obtained from $K_{3,3}$ by a finite (possibly empty) sequence of vertex splits.

In this section, we will prove Theorem 1.4, the Petersen family graphs are the MMNA graphs with $\|G\| \leq 16$. Recall that the Petersen family is the set of seven graphs obtained by $\nabla\mathrm{Y}$ and $\mathrm{Y}\nabla$ moves on the $(10, 15)$ Petersen graph P_{10}. In addition to P_{10}, the set includes K_6, $K_{3,3,1}$, $K_{4,4} - e$, and, by definition, is closed under $\nabla\mathrm{Y}$ and $\mathrm{Y}\nabla$ moves. We first observe that each graph in the family is MMNA.

Lemma 2.3. *The seven graphs in the Petersen family are all MMNA*

Proof. Aside from describing what is to be checked, we omit most of the details. Let G be a graph in the Petersen family. It's enough to verify that $\forall v \in V(G)$, $G - v$ is non-planar and that $\forall e \in E(G)$, deletion and contraction of e both result in apex graphs. □

The proof of Theorem 1.4 depends on the following lemma that characterises NA graphs using the idea of a vertex near a branch vertex. If G is a graph and $w \in V(G)$ is such that there is a path from w to a branch vertex, a, of G that contains no other branch vertices of G, then we say w is **near** a. Similarly, if w is a vertex in some $G + v$, w is near a branch vertex a of G if there is a w-a path independent of the other branch vertices.

Lemma 2.4. *Suppose* G *simplifies to* K_5 *or* $K_{3,3}$*. Then* $G + v$ *is NA if and only if* v *is near every branch vertex of* G*.*

Proof. As in the definition above, forming G^s, the simplification of G, determines a set of branch vertices.

First, assume that $G+v$ is NA and v is not near a branch vertex of G, call it a. If we remove a branch vertex near a, call it b, then, we claim, $G+v-b$ is planar, which contradicts $G+v$ NA. To verify the claim, note that G^s is minor minimal non-planar. The only way that $G+v-b$ could be non-planar would be for v to take the place of b in that graph. This would require independent paths from v to each of the branch vertices near b. As there is no such v-a path, $G+v-b$ is planar.

Now assume that, in $G+v$, v is near every branch vertex of G. Then $G^* = (G+v)^s$ is of the form $H+v$ where H is a subdivision of G^s and, by abuse of notation, we again refer to the vertices of H of degree three or four as branch vertices (of G). In G^*, the neighbors of v are either branch vertices of G or on edges of G^s that were subdivided to form H. In particular, v is near the same branch vertices in $H+v$ as it was in $G+v$. We wish to show that G^* can, through a series of Y∇ moves, be transformed into an NA graph. If, in G^*, v is adjacent to all the branch vertices of G, we are done, since if $G^s = K_5$, then G^* has a K_6 minor, and if $G^s = K_{3,3}$ then G^* has $K_{3,3,1}$ as a minor. As K_6 and $K_{3,3,1}$ are both NA (see previous lemma), $G+v$ is as well.

Next, choose a branch vertex from G, call it a. Suppose v is not adjacent to a in G^*. However, we've assumed v is near every branch vertex, including a. Hence there is a vertex of degree three that has both a and v as neighbors, call it w. Performing a Y∇ move on w makes a and v neighbors and will not change the nearness of v with any branch vertices. Repeating this process for the rest of the branch vertices results in a graph where v is adjacent to each branch vertex of G. Again, if $G^s = K_5$, then this series of Y∇ moves on G^* gives a graph that has a K_6 minor. If $G^s = K_{3,3}$ then a series of Y∇ moves on G^* gives us a graph that has $K_{3,3,1}$ as a minor. Since Y∇ and ∇Y preserve the Petersen family, we conclude that $G+v$ has a minor from the Petersen family and is, therefore, NA. □

The proof shows that, not only is $G+v$ NA, it has a Petersen family graph as a minor. On the other hand, if $G+v$ has a Petersen family graph minor, then it is NA by Lemma 2.3. Also, Petersen family graph minors characterize intrinsic linking [RST]. The following lemma combines these observations.

Lemma 2.5. *Let G be a graph with vertex v such that $(G-v)^s = K_5$ or $K_{3,3}$. Then the following are equivalent.*

- *The vertex v is near every branch vertex of $G-v$.*
- *G is NA.*
- *G has a Petersen family graph minor.*
- *G is intrinsically linked.*

Lemma 2.6. *If $G+a$ is formed by adding a degree three vertex a to a split $K_{3,3}$ graph G and $G+a$ is NA, then $(G+a)^s$ is the Petersen graph.*

Proof. By Lemma 2.4, there are paths from a to each branch vertex that avoid all other branch vertices. Up to isomorphism, the only way to arrange this is as in the graph of Figure 2, which is the Petersen graph. □

Figure 2 illustrates the idea of a vertex being near an edge. Let G be such that $G^s = K_{3,3}$ or K_5. As in the proof of Lemma 2.4, if we add a vertex v, then, in

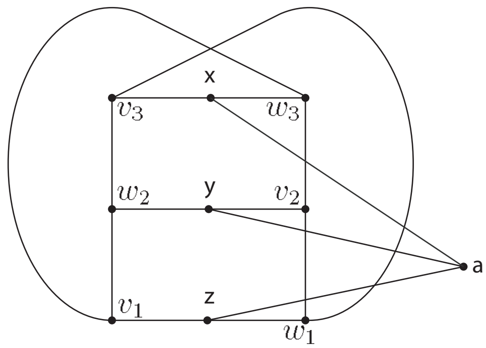

FIGURE 2. Adding a degree 3 vertex to a split $K_{3,3}$ yields the Petersen graph.

general, $(G+v)^s$ will be of the form $H+v$ where H is a subdivision of G^s. We say that v is **near the edge** xy in G^s, where x and y are branch vertices, if, in $(G+v)^s$, v has a neighbor interior to the (subdivided) edge xy of G^s. In Figure 2, a is near the edges v_iw_i, $i = 1, 2, 3$.

Lemma 2.7. *If $G+a$ is formed by adding a vertex a of degree four to a split $K_{3,3}$ graph G and $G+a$ is NA, then $(G+a)^s$ is one of the seven graphs in Figure 3.*

Proof. By Lemma 2.4, there are paths from a to each branch vertex that avoid all other branch vertices. Let $N(a) = \{n_1, n_2, n_3, n_4\}$. As there are six vertices and $d(a) = 4$, then there is an n_i, say n_1, that has an edge, say v_1w_1, as its nearest part. Since there are four branch vertices left and three neighbors of a, another n_i, say n_2, must have an edge as its nearest part with vertices disjoint from $\{v_1, w_1\}$, call it v_2w_2. There are three graphs generated when a has a neighbor whose nearest part is a branch vertex of G and four more when a has no such neighbor. Figure 3 shows the graphs that results from this condition. □

We conclude this section with a proof of Theorem 1.4. The proof requires one additional lemma. Let $\delta(G)$ and $\Delta(G)$ denote the **minimum** and **maximum degree** of graph G.

Lemma 2.8. *Suppose G has $\delta(G) = 3$, $\Delta(G) = 4$, and $13 \leq \|G\| \leq 16$. Then either there is a degree 4 vertex with a degree 3 neighbor or else G is the disjoint union $K_5 \sqcup K_4$.*

Proof. For a contradiction, suppose no degree 4 vertex has a degree 3 neighbor. Then G is disconnected with cubic and quartic components. The smallest quartic graph is K_5 with ten edges and the smallest cubic graph is K_4 with six. So, the order of G is at least 16 and $K_5 \sqcup K_4$ is the only way to realize that minimum. □

Proof. (of Theorem 1.4) As stated in Lemma 2.3, the Petersen family graphs are all MMNA. What is left is to show that they are the only such graphs on 16 or fewer edges. Suppose G is an MMNA graph with 16 or fewer edges and suppose that it is not in the Petersen family. If $\delta(G) < 3$, then contracting an edge of a vertex of small degree or deleting an isolated vertex results in a proper minor that is still NA, contradicting minor minimality. So we assume $\delta(G) \geq 3$.

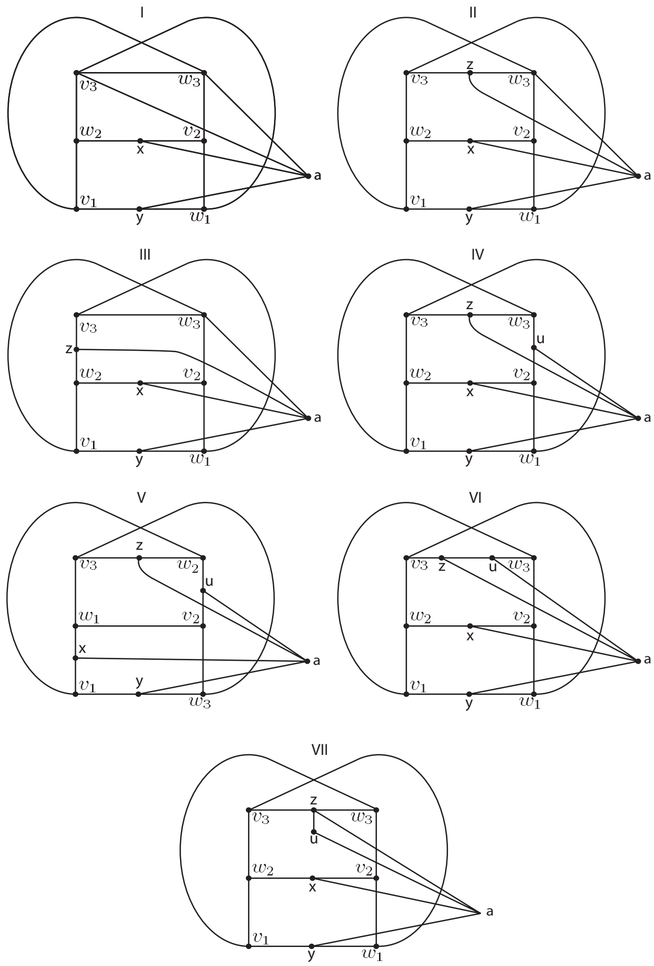

FIGURE 3. Adding a degree 4 vertex to a split $K_{3,3}$.

Then, since a non-planar graph has at least nine edges, G must have at least 12 edges. If $\|G\| = 12$, it must be cubic. But, then, removing a vertex a results in $\|(G-a)^s\| = 6$ so that $G-a$ is planar and G is apex, a contradiction. So we can assume $\|G\| \geq 13$.

Similarly, if G has 13 edges, then G cannot have a vertex of degree five or more, lest $G-a$ be non-planar. On the other hand, G is certainly not cubic, so, by Lemma 2.8, there is a degree 4 vertex a that has a degree 3 neighbor. Again, $\|(G-a)^s\| \leq 8$, so that $G-a$ is planar, a contradiction. We can assume $\|G\| \geq 14$.

Suppose G has 14 edges. If G contains a degree 5 vertex a, then $G-a$ must be $K_{3,3}$. By Lemma 2.4, G cannot be NA. Suppose there's a degree 4 vertex a having a degree 3 neighbor. Then $\|(G-a)^s\| \leq 9$, so $(G-a)^s = K_{3,3}$, as otherwise, G is apex. This also means that $G-a$ is $K_{3,3}$ with a single edge subdivision. By Lemma 2.7, G is not NA, a contradiction.

Having 14 edges, G is not cubic and we've argued that there can be no degree 4 vertex with a degree 3 neighbor. So, by Lemma 2.8, G must be quartic. Then deleting any vertex a results in a $(6, 10)$ graph. If G is NA, $G - a = K_{3,3} + e$ and since G was 4-regular, the $N(a)$ is exactly the degree three vertices in $K_{3,3} + e$. However, G is then apex. We conclude G must have at least 15 edges.

If G has 15 edges, then $\Delta(G) \leq 6$ since a non-planar graph has at least nine edges. By Lemma 2.5, if $(G - a)^s$ is $K_{3,3}$ or K_5, then G is NA if and only if it has a minor from the Petersen family. Hence, if G is an MMNA 15 edge graph, finding a vertex whose removal induces a graph that simplifies to $K_{3,3}$ or K_5 implies that G is a member of the Petersen family. In particular, if G is cubic (see Lemma 2.6), or has a a vertex of degree 6, then G is a member of the Petersen family.

Let us assume that $\Delta(G) = 4$. Since there are no quartic graphs of 15 edges, by Lemma 2.8, there is a degree 4 vertex a, with at least one neighbor of degree 3. If a has more than one neighbor of degree 3 or if $G - a$ is a subdivision of K_5 or $K_{3,3}$ then, by Lemma 2.5, we are done. In particular, if a has more than one neighbor of degree 3, then $\|(G-a)^s\| \leq 9$. However, as $(G-a)^s$ must be non-planar, $(G - a)^s = K_{3,3}$ and we are done.

So we can assume a has exactly one degree 3 neighbor and that $G - a$ is a subdivision of a 10 edge non-planar graph other than K_5. Then this graph is either the simple graph $K_{3,3} + e$ or the multigraph formed by doubling a single edge of $K_{3,3}$ (see Figure 13).

If $G - a$ is the multigraph, then $(G - a)^s = K_{3,3}$ and we can apply Lemma 2.5. So, suppose $G - a$ is formed by subdividing an edge of $K_{3,3} + e$ (see Figure 13b). If the subdivision is not on the added v_2v_3 edge, then $G - w_3$ is planar. This is because a is not adjacent to either v_3 or v_2 and there is only one additional vertex from subdivision in forming $G - a$. So it must be v_2v_3 that is subdivided to form $G - a$. This means $\{w_1, w_2, w_3\} \subset N(a)$ as otherwise v_2 will have two neighbors of degree 3. The resulting graph is $K_{4,4} - e$, a member of the Petersen family.

So, we can assume $\Delta(G) = 5$. Let a be a degree 5 vertex. Then, being non-planar, $(G-a)^s$ has at least nine edges. If $(G-a)^s$ is K_5 or $K_{3,3}$, Lemma 2.5 implies that we are done. So $G - a$ must be $K_{3,3} + e$ shown in Figure 13b. Since a is of degree 5 it must be adjacent to either w_2 or w_3, say w_3. Then, a must be adjacent to both v_3 and v_2, as otherwise $G - w_3$ is planar. However, a must then also be adjacent to w_1 and w_2. If not, v_2 is a degree 5 vertex with a degree 3 neighbor, meaning $G - v_2$ is planar, a contradiction. Thus, $N(a) = \{v_2, v_3, w_1, w_2, w_3\}$ and the resulting graph is the $(7, 15)$ Petersen family graph that comes from a ∇Y move on K_6. We call this graph P_7.

Next suppose $\|G\| = 16$. We can assume $\Delta(G) \leq 6$. Indeed, if $\Delta(G) \geq 8$, there's a vertex a whose deletion gives $G - a$ of size at most eight, hence planar. If $\Delta(G) = 7$, deleting a degree 7 vertex a means $\|G - a\| = 9$. As $G - a$ must be non-planar, it is $K_{3,3}$ and we can apply Lemma 2.5.

Suppose $\Delta(G) = 6$ and let a be a degree 6 vertex. Then $G - a$ is a non-planar graph of size 10 and minimal degree at least two. If $G-a$ is K_5, we apply Lemma 2.5, so we can assume $G - a$ is $K_{3,3} + e$ (see Figure 13b). Since a has degree 6 in G, it is adjacent to all vertices of $K_{3,3} + e$ so that G has the Petersen family graph $K_{3,3,1}$ as a subgraph.

If $\Delta(G) = 5$, let a be a vertex of top degree. There are two cases depending on whether or not a has a degree 3 neighbor. If so, $\|(G - a)^s\| \leq 10$. By assumption,

$(G-a)^s$ is non-planar and, if $(G-a)^s = K_5$ or $K_{3,3}$, we can apply Lemma 2.5. So we may assume that $(G-a)^s$ is the graph $K_{3,3}+e$ (see Figure 13b) and $G-a$ is formed by subdividing a single edge of that graph. If the subdivided edge is the added edge v_2v_3, then $w_3 \in N(a)$ as otherwise, $G - v_3$ is planar. By symmetry $w_1, w_2 \in N(a)$ as well and G has the Petersen family graph $K_{4,4} - e$ as a subgraph. So, we can assume that it is not v_2v_3 that is subdivided.

Suppose it is some other edge incident to v_2 or v_3, say v_3w_3, that is subdivided. Then $G - w_3$ is planar unless v_2 and v_3 are both neighbors of a. But in that case, there will be a degree 5 vertex b with at least two degree 3 neighbors. This means $\|(G-b)^s\| \leq 9$, so it is either planar, a contradiction, or $K_{3,3}$ and we can apply Lemma 2.5. Thus, the subdivided edge is adjacent to neither v_2 nor v_3. Without loss of generality, it is v_1w_1 that is split to create $G - a$. Still, $G - w_3$ is planar unless $v_2, v_3 \in N(a)$ and again we will be left with a degree 5 vertex with at least two degree 3 neighbors.

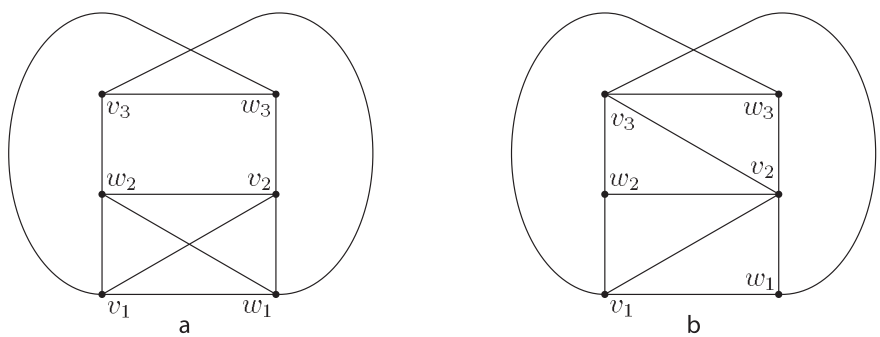

FIGURE 4. Non-planar $(6,11)$ graphs with $\delta(G) \geq 3$.

So, we can assume a has no degree 3 neighbor. Then $G - a$ is non-planar, of size 11, and minimal degree three. The only possibilities are the $(6,11)$ graphs of Figure 4 or the $(7,11)$ graph of Figure 16ii. We can assume that no degree 5 vertices have a degree 3 neighbor in G as otherwise we return to the previous case. Suppose first that $G-a$ is the $(6,11)$ graph of Figure 4a. Then $N(a)$ must include v_3 and w_3, the degree 3 vertices of $G-a$ as otherwise there'll be a degree 5 vertex with a degree 3 neighbor. Without loss of generality, w_1 is the vertex of $G-a$ missing from $N(a)$. Then $G - v_1$ is planar, a contradiction. Similarly, if $G-a$ is the $(6,11)$ graph of Figure 4b, then, since we assumed $\Delta(G) = 5$, it's v_2 that is missing from $N(a)$, in which case $G - w_2$ is planar. Finally, suppose $G-a$ is the $(7,11)$ graph of Figure 16ii. We see that $v_2 \in N(a)$ as otherwise, $G - w_3$ is planar. But then v_2 is a degree 5 vertex in G and can have no degree 3 neighbors. Thus $N(a) = \{u, v_2, w_1, w_2, w_3\}$ and contracting uv_1 gives the Petersen family graph P_7 as a minor. (Recall that P_7 is the result of a ∇Y move on K_6.)

Next assume $\Delta(G) = 4$. If G is quartic, it is one of the six quartic graphs of order eight. Only two of these are NA. One is $K_{4,4}$, which has the Petersen family graph $K_{4,4} - e$ as a subgraph. The other comes from splitting the degree 6 vertex of the Petersen family graph $K_{3,3,1}$. Thus, we can assume $\delta(G) = 3$ and, by Lemma 2.8, there is a degree 4 vertex a with a degree 3 neighbor. Then $\|(G-a)^s\| \leq 11$. By

Lemma 2.5, $(G-a)^s$ is of size 10 at least, so we can assume each degree 4 vertex has at most two degree 3 neighbors.

Suppose then that $\|(G-a)^s\| = 10$ meaning $G-a$ is formed by making two edge subdivisions on $K_{3,3}+e$ (Figure 13b). Suppose further that neither of the subdivisions occur on the added edge v_2v_3. Then $G-w_3$ is planar unless the subdivisions are on the edges v_2w_2 and v_3w_2 (or v_2w_1 and w_3w_1, a case we can omit due to symmetry.) If these are the subdivisions, then $w_3 \in N(a)$ as otherwise $G-v_3$ is planar. Finally, deleting the vertex on v_2w_2 formed by the subdivision, call it u, gives a planar graph unless $w_1 \in N(a)$. So, we can assume a is adjacent to u, w_1, and w_3 as well as the vertex formed by subdividing v_3w_2. Then, contracting uw_2 leads to the $(8,15)$ Petersen family graph resulting from two Y∇ moves on the Petersen graph. We call this $(8,15)$ graph P_8. So, assuming there is no subdivision on v_2v_3 leads to a graph with a Petersen family graph minor.

Thus, we can assume there is at least one subdivision on v_2v_3. This means that v_2 and v_3 already have one degree 3 neighbor. Since they may have at most two, then two of w_1, w_2, and w_3, say the last two, are adjacent to a. In order that $G-w_2$ and $G-w_3$ are both non-planar, the final neighbor of a, call it u, arises by subdivision of an edge incident to w_1. Then contracting uw_1 shows that G has the Petersen family graph $K_{4,4}-e$ as a minor. So, we can assume $\|(G-a)^s\| \geq 11$.

Since $\delta(G)=3$, then $|G| \geq 9$. So, if $(G-a)^s$ has size 11, then it has at least order seven. Thus, $(G-a)^s$ is the $(7,11)$ graph of Figure 16ii and $G-a$ is formed by a single subdivision. Also, we may assume every degree 4 vertex has at most one degree 3 neighbor (as otherwise we return to the previous case). So that both $G-w_2$ and $G-w_3$ are non-planar, the subdivision must be of uv_2 or v_2w_1. Either way, this constitutes a degree 3 neighbor of v_2 and its remaining neighbors must all be adjacent to a. However, in both cases, this results in a degree 4 vertex (e.g., w_1 or u, respectively) with two degree 3 neighbors, which puts us back in the previous case. This completes the argument in the case $\|G\| = 16$ and with it the proof. $\square$

3. 14 VERTEX GRAPHS

In this section we show the following (originally proved in [BM]):

Proposition 3.1. *If G is a $(14,21)$ MMN2A graph, then G is in the Heawood family.*

Proof. Let G be a $(14,21)$ MMN2A graph. We can assume $\delta(G) \geq 3$ as otherwise a vertex deletion or edge contraction on a small degree vertex will give a proper minor that is also N2A. Then G must have the degree sequence (3^{14}) and for any $a \in V(G)$, $G-a$ has the sequence $(3^{10}, 2^3)$. Now choose another vertex, b, such that $G^* = G-a, b$ has the sequence $(3^6, 2^6)$ (i.e., a and b have no common neighbors). There are enough degree 3 vertices in $G-a$ to assure we can always choose such a b.

Since G is N2A and G^* has the sequence $(3^6, 2^6)$, then G^* must be a split $K_{3,3}$. By Lemma 2.6, $(G^*+a)^s$ is the Petersen graph. Then $G' = (G^*+a)-w_3$ is another split $K_{3,3}$.

By Lemma 2.4, b must have a path to a that avoids v_3, w_1, w_2, y, and z. Since a and b have no common neighbors, this means b has a neighbor b_1 that is adjacent to x. So, there are two cases: in $G'+b$, either b_1 is of degree two, or else it has v_3 as a third neighbor. (See Figure 5.)

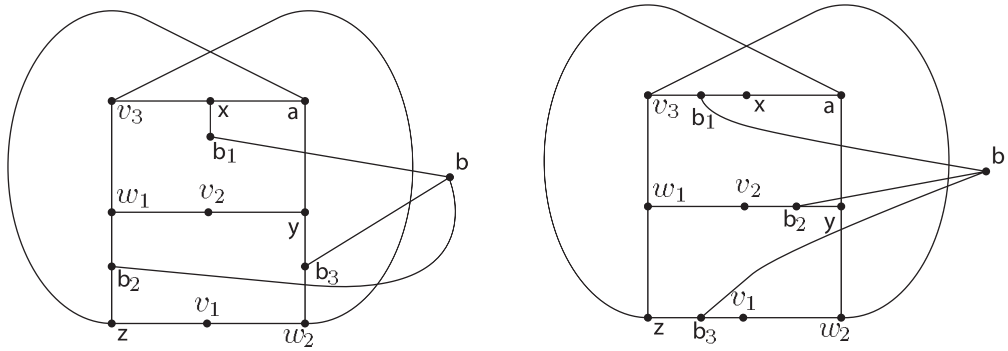

FIGURE 5. Two possibilities for $G' + b$.

In either case, b_1 gives paths from b to the branch vertices a and v_3 and there are three ways to split the remaining four branch vertices into two pairs. However, we see that $G - w_2, z$ is planar (and G is 2–apex), unless we make the choices shown in Figure 5. In both cases, adding w_3 back will give us the Heawood graph. Hence the only (14,21) MMN2A graph is the Heawood graph, which is in the Heawood family. □

4. 13 VERTEX GRAPHS

In this section we prove the following:

Proposition 4.1. *If G is a $(13, 21)$ MMN2A graph, then G is in the Heawood family.*

Proof. Let G be a MMN2A $(13, 21)$ graph. Consider the degree sequences $(3^{12}, 6)$ and $(3^{11}, 4, 5)$. If we remove the vertex of highest degree the resulting graph simplifies to a graph with fewer than 14 edges, hence (by Theorem 1.4) to an apex graph. So G does not have such a degree sequence.

Then G has the sequence $(3^{10}, 4^3)$. Again, if a is a vertex of degree 4 that has three neighbors of degree 3, then $(G - a)^s$ is apex, so this cannot be the case. We conclude that the degree 4 vertices form a triangle in G and that there is a degree 3 vertex in G, call it a, whose neighbors all have degree 3. This means that $G - a$ simplifies to a graph $G^* = (G - a)^s$ with degree sequence $(3^6, 4^3)$. Since G^* must be NA, and has 15 edges, by Theorem 1.4 it is in the Petersen family. There is a unique nine vertex graph in the family, which we call P_9, see Figure 6.

Note that in Figure 6 there is a unique triangle, which we'll denote xyz and label the corresponding vertices in $G - a$ and G as x, y, and z as well. Notice also that x, y and z all have degree 4 in G^* so none of them are neighbors of a in G. Moreover, we assumed x, y and z form a triangle in G, and since the triangle is clearly preserved in G^*, it must also be preserved in $G - a$. In particular, this implies that a is not near any of the edges that form this triangle, i.e., none of the degree 2 vertices deleted in simplifying from $G - a$ to G^* are on the edges of the triangle.

Observe that $(G - a, y)^s = K_{3,3}$ and that the induced graph after adding a back must be NA. Hence, by Lemma 2.4, a must have a path to each branch vertex that

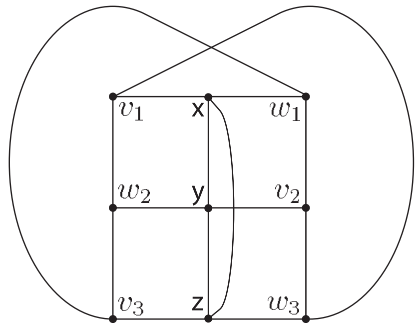

FIGURE 6. The Petersen family graph P_9.

does not go through any other branch vertex. Since a is not near the edge xz, it must be near either edges xw_1 or xv_1 and zw_3 or zv_3. Similarly, $(G-a,x)^s$ shows that a must also be near yw_2 or yv_2.

We claim that a is near xw_1, yw_2, and zw_3 or xv_1, yv_2, and zv_3, in which case G is the Heawood family graph C_{13}. (See [HNTY] for the names, like C_{13}, of the Heawood family graphs. This is the unique order 13 graph in the Heawood family and corresponds to graph 15 in Figure 1). Otherwise, either a is near xv_1 and yw_2 or xw_1 and yv_2, in which case $G - v_3, w_3$ is planar, or else a is near zv_3 and yw_2 or zw_3 and yv_2 in which case $G - v_1, w_1$ is planar. Therefore the proposition is proved. □

5. 12 VERTEX GRAPHS

In this section we prove that a $(12,21)$ MMN2A graph G is in the Heawood family. This means G is one of three graphs that are called H_{12}, C_{12}, and N'_{12} by Hanaki et al. [HNTY] and are represented as graphs 12, 13, and 19, respectively, in Figure 1. We first observe that if G is triangle-free and of the correct degree sequence, it must be H_{12}. This was originally proved in [BM].

Lemma 5.1. *Let G be MMN2A of degree sequence $(3^6, 4^6)$ and triangle free. Then G is H_{12}.*

Proof. Note that if any of the vertices of degree 4 have three or more neighbors of degree 3, removing such a vertex results in an apex graph by Theorem 1.4, so we may assume this doesn't happen. We also notice that we can either single out a degree 3 vertex, all of whose neighbors are degree 3 vertices, or a degree 4 vertex that has two degree 3 neighbors. To see this, suppose it is not the case. Since G has no triangles, the subgraph induced by the degree 4 vertices is $K_{3,3}$ and each of the vertices has a unique neighbor of degree 3. Hence, removing two non-adjacent vertices of degree 4 results in a graph that simplifies to a graph of size eight, thus planar. Hence G would not be 2-apex.

Now assume that we do not have a vertex of degree 4 with two degree 3 neighbors. Say that a is a degree 3 vertex whose neighbors are all of degree 3. Then $(G-a)^s$ has degree sequence $(3^2, 4^6)$. Theorem 1.4 implies that it is $K_{4,4} - e$. Because G has no degree 4 vertex with two degree 3 neighbors, we know that the edge subdivisions from $(G-a)^s$ to $G-a$ are all on edges incident to the degree 3 vertices of $(G-a)^s$.

Since there are exactly three subdivisions from $(G-a)^s$ to $G-a$, there is one vertex of degree 3 in $(G-a)^s$ that gets at least two subdivisions, call it a_1. So, a_1 has degree 4 neighbors v_1, v_2 in $(G-a)^s$ so that a_1v_1 and a_2v_2 are subdivided in forming $(G-a)$. Then $G - v_1, v_2$ is planar and G is 2-apex.

So we may assume that a has degree 4 and there exist $b, c \in N(a)$ such that $d(b) = d(c) = 3$ and $c \neq b$. Then $(G-a)^s$ has degree sequence $(3^6, 4^3)$ which tells us, by Theorem 1.4, that it is P_9. Furthermore, since G does not have a triangle, we know that one of the subdivisions from $(G-a)^s$ to $G-a$ is on the triangle xyz of Figure 6; say it's xy that is subdivided. Removing either x or y, Lemma 2.4 tells us that the other subdivision from $(G-a)^s$ to $G-a$ must be on an edge incident to z. We may say it is the edge yz without losing generality. Now, remove y and it is easy to see that Lemma 2.4 forces a to be adjacent to w_2 and v_2. Therefore G is H_{12}. $\square$

Proposition 5.2. *If G is a $(12, 21)$ MMN2A graph, then G is in the Heawood family.*

Proof. We assume again that G is MMN2A and that G is a $(12, 21)$ graph. We can assume the maximum degree $\Delta(G)$ is at most five. For a vertex a with $d(a) \geq 6$ in a $(12, 21)$ graph with $\delta(G) \geq 3$ will have at least one neighbor of degree 3. Then $(G-a)^s$ has at most 14 edges and is apex, by Theorem 1.4. This implies $G-a$ is apex and G is 2–apex, a contradiction.

This leaves four possible degree sequences: $(3^9, 5^3)$, $(3^8, 4^2, 5^2)$, $(3^7, 4^4, 5)$, and $(3^6, 4^6)$.

Let G have the degree sequence $(3^9, 5^3)$ or $(3^8, 4^2, 5^2)$. Then any a with $d(a) = 5$ has at least two neighbors of degree 3. This means $(G-a)^s$ simplifies to a graph with fewer than 15 edges and so it is apex (Theorem 1.4), whence G is 2–apex, a contradiction.

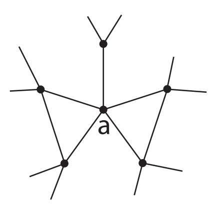

FIGURE 7. Graph near the degree 5 vertex a.

We now focus our attention on the case where G has the degree sequence $(3^7, 4^4, 5)$ and show that the only MMN2A graph with this degree sequence is C_{12}. (See [HNTY] for the name. This is graph 12 in Figure 1.) Let a denote the vertex of degree 5. Note that a has at most one neighbor of degree 3, as otherwise $\|(G-a)^s\| \leq 14$ meaning $G-a$ is apex (Theorem 1.4) and G is 2–apex. Hence, the neighbors of a are all the vertices of degree 4 and one vertex of degree 3. Moreover, each vertex of degree 4 has at most 2 neighbors of degree 3. This is illustrated in Figure 7 . This implies that $(G-a)^s$ is a NA 3-regular graph with 15 edges, i.e., the Petersen graph (see Figure 2). Since the Petersen graph has no triangles or 4-cycles, we see that $G-a$ has no four cycles. This implies that the vertices

of degree 4 do not form a triangle or 4-cycle in G. This justifies the specifics of Figure 7.

Let $b \in V(G)$ denote one of the vertices of degree 4. From the above paragraph, we argued that b must have exactly two neighbors of degree 3, hence $(G-b)^s$ is a $(9, 15)$ graph with degree sequence $(3^6, 4^3)$. This implies that $(G-b)^s$ is the Petersen family graph P_9 illustrated in Figure 6 (the unique Petersen family graph on nine vertices). In $G - b$, vertex a has degree 4 and without loss of generality is vertex y in the figure. We have deduced that b is adjacent to a as well as either w_2 or v_2, say v_2. Note that b is not near the edge xz. In order for $G - a$ to be NA, by Lemma 2.4, b is near the edges v_1x and v_3z. Adding both a and b back in shows that this graph is C_{12}.

Now let G have the degree sequence $(3^6, 4^6)$. We will show G is either H_{12} or else N'_{12}. (See [HNTY] for these names. There are graphs 12 and 19 respectively in Figure 1.) By Lemma 5.1, the only triangle free MMN2A graph with degree sequence $(3^6, 4^6)$ is H_{12}, so we will assume that G has a triangle and show that this implies it is N'_{12}. By Theorem 1.4, each degree four vertex in G can have at most two neighbors of degree 3. Notice that in N'_{12}, each degree 4 vertex has exactly one neighbor of degree 3 and vice versa. We argue that G must also share this property in order to be MMN2A.

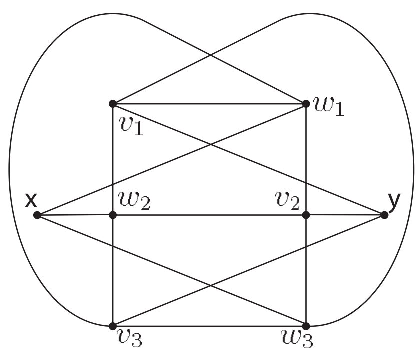

FIGURE 8. The Petersen family graph $K_{4,4} - e$.

First, assume there is an $a \in V(G)$ such that a has degree 3 and three degree 3 neighbors. Hence $G^* = (G-a)^s$ has degree sequence $(4^6, 3^2)$ and is an $(8, 15)$ graph. Since G being MMN2A implies that G^* is NA, by Theorem 1.4 it is in the Petersen family. By the degree sequence $(4^6, 3^2)$, we can identify G^* as $K_{4,4} - e$ drawn in Figure 8. Since G^* has no triangles, the triangle of G is formed in reattaching a. Hence there is at least one edge in G^* that is subdivided twice in returning to $G - a$. Because of the symmetry of G^*, we may assume without loss of generality that these subdivisions are on the edges v_1w_1 or yv_1. In the first case $G - v_1, w_1$ is planar and the second splits into two cases: either the other subdivsion from G^* to $G-a$ occurs on an edge incident to x in G^* or it does not. In the case where it does not, then $G-v_i, w_j$ is planar, where v_i and w_j are the vertices in G^* between which the subdivision occurs or v_1w_1 if it's on an edge incident to y. In the other case, $G - x, v_1$ is planar since it is essentially the same as the planar graph $G^* - x, v_1$

with an extra path from y to a w_i . So, in an MMN2A graph, every degree 3 vertex has at least one degree 4 neighbor.

Now suppose $a \in V(G)$ is a degree 4 vertex with exactly two neighbors of degree 3. Then $G^* = (G-a)^s$ has degree sequence $(4^3, 3^6)$. Since G^* must be NA, by Theorem 1.4 it is in the Petersen family and hence is the graph P_9 shown in Figure 6. In the following, we use the labeling of that figure.

When we remove x, y, or z separately from G^* each induced subgraph shows us (by Lemma 2.4) that a must have paths to x, y and z in G that do not include any of their neighbors in G^*. As these three vertices already have degree 4, the neighborhood of a includes vertices adjacent to x, y, z created by edge subdivisions.

Since there are only two edge subdivisions from G^* to $G-a$, this implies that one has to be on the xyz triangle. By the symmetry of G^* we can assume without loss of generality that xy is subdivided. The other subdivision is on an edge incident to z in G^*. Since we assume that G contains a triangle, a must be part of that triangle. Observe that $(G^* - y)^s = K_{3,3}$. By Lemma 2.4, a must have paths to the vertices v_1 v_3, w_1, w_3, x, and z in $G-y$ that exclude the others from that list. Now, a is adjacent to exactly two vertices in $G^* - y$ (as the two other neighbors appear only after additional edge subdivisions) and since we have already established that a is near both x and z and possibly v_3 or w_3, the remaining neighbors of a are either w_2 and v_2, v_1 and v_2, or w_1 and w_2. Recalling that a is not actually adjacent to x, just simply near it by way of a subdivison of xy in G^*, and since G must have a triangle, none of these cases can be G.

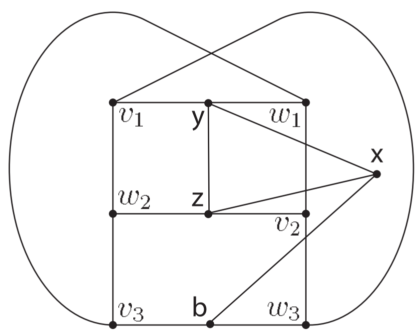

FIGURE 9. Graph after removing a degree 4 vertex leaving a triangle.

To summarize, we established that if G is MMN2A with degree sequence $(3^6, 4^6)$ and contains a triangle, then each vertex of degree 4 has at most one neighbor of degree 3 and each vertex of degree 3 has at least one neighbor of degree 4. Hence, there is a one to one correspondence between the degree 4 vertices and the degree 3 vertices by the relation of being neighbors in G. Note that none of the degree 3 vertices can be part of a triangle in G, otherwise, it would either be adjacent to at least two degree 4 vertices or else there is a degree 4 vertex with two neighbors of degree 3. Thus, we can assume there is a triangle of vertices of degree 4 in G. Choose some vertex of degree 4 not on this triangle, call it a. Then $G^* = (G-a)^s$ has degree sequence $(3^8, 4^2)$ and contains a triangle. We claim that G^* is the graph illustrated in Figure 9. Note that the two degree 4 vertices in G^* are adjacent. So, if we delete one of them, denote it b, then $(G^* - b)^s$ has nine edges and must

be non-planar since G^* is NA. Thus $(G^* - b)^s = K_{3,3}$ and, using Lemma 2.7, and that G^* has a triangle and degree sequence $(3^8, 4^2)$, we deduce G^* is as shown in Figure 9.

Now that we have established what G^* looks like (Figure 9), determining where a goes is easy. For starters, since both y and z are adjacent to x, then x cannot have degree 3 due to the one to one correspondence between vertices of degree 3 and 4. So a is adjacent to x. Either a is adjacent to v_1 or w_1 since y is adjacent to only one vertex of degree 3, say w_1. Then, for the same reason x and a were adjacent, a and v_2 are adjacent. Since $G - z$ is NA, by Lemma 2.4, a is near w_2v_3 or v_1w_2. Similarly, $G - y$ is NA and Lemma 2.4 shows a is near v_1w_2 or v_1w_3. So a is near v_1w_2. This graph is N'_{12}. Therefore, the only graph MMN2A graph with degree sequence $(3^6, 4^6)$ that contains a triangle is N'_{12}. □

6. 11 vertex graphs

In this section we prove that an $(11, 21)$ MMN2A graph is in the Heawood family. We begin with five lemmas, one each for the Heawood family graphs of this order: E_{11}, C_{11}, H_{11}, N'_{11}, and N_{11}. (See [HNTY] for the names. These correspond to graphs 8, 10, 11, 16, and 17 respectively in Figure 1.)

Lemma 6.1. *Let G be an $(11, 21)$ MMN2A graph with degree sequence $(3^4, 4^6, 6)$. Then G is C_{11}.*

Proof. Consider $b \in V(G)$ such that $\deg(b) = 6$. Notice that for any $v \in N(b)$ we must have $\deg(v) = 4$, otherwise (Theorem 1.4) $G - b$ is not NA. This implies that $G - b$ must be the Petersen graph (see Figure 2). Without loss of generality, we can assume that the vertex a in Figure 2 is not a neighbor of b in G. Since $(G - b, x)^s = K_{3,3}$, then in $G - x$, by Lemma 2.4, b must be adjacent to z and y. Similarly, if we consider $G - b, z$ we see that b is adjacent to x. Consider again $G - x$. Since b has degree 5 in $G - x$, is adjacent to y and z, and must have paths to v_1, v_2, w_1, and w_2 that do not go through v_1, v_2, w_1, w_2, x, or y, we see that b is adjacent to either v_3 or w_2 or both. Similarly, considering $G - y$ and $G - z$, we see that b is adjacent to either v_2 or w_2 and v_1 or w_1. We claim that b is adjacent to v_1, v_2, and v_3 or w_1, w_2, and w_3 in which case we have C_{11}. Otherwise, if $v_2 \in N(b)$ and $w_1 \in N(b)$ then $G - v_3, w_3$ is planar, or if $v_2 \in N(b)$ and $w_3 \in N(b)$ then $G - w_1, v_1$ is planar. Similarly, if $w_2 \in N(b)$ and $v_1 \in N(b)$ then $G - v_3, w_3$ is planar, or if $w_2 \in N(b)$ and $v_3 \in N(b)$ then $G - w_1, v_1$ is planar. Therefore G must be C_{11}. □

Lemma 6.2. *Let G be an $(11, 21)$ MMN2A graph with degree sequence $(3^5, 4^3, 5^3)$. Then G is E_{11}.*

Proof. We may assume that $\exists a \in V(G)$ such that $\deg(a) = 5$ and $\exists u \in N(a)$ such that $\deg(u) = 3$. If not, then removing any two of the degree 4 vertices results in a K_4 graph with a bridge to a graph of at most seven edges, which is clearly planar. So we may assume that $G^* = (G - a)^s$ has degree sequence $(3^6, 4^3)$. This means that G^* is the Petersen family graph P_9 shown in Figure 6. By the degree sequence of the original G, we may assume, without loss of generality, that a is adjacent to x and y (referring again to Figure 6), and hence is not adjacent to z. Removing either x or y, Lemma 2.4 shows us that a is near an edge incident to z. If a is near the edge yz or xz, then a is also adjacent to two more vertices in Figure 6. Removing both of these results in a planar graph. Thus a is near the edge v_3z or the edge w_3z. By symmetry, we will assume v_3z.

Applying Lemma 2.4 to $G-y$ shows that a must be adjacent to v_2 and, similarly, considering $G-x$ shows us that a must be adjacent to v_1. Reassembling G gives E_{11}. □

Lemma 6.3. *Let G be an $(11, 21)$ MMN2A graph with degree sequence $(3^4, 4^5, 5^2)$. Then G is H_{11}.*

Proof. Assume that $\exists a \in V(G)$ such that $\deg(a) = 5$ and $\exists u \in N(a)$ such that $\deg(u) = 3$. Then $G^* = (G-a)^s$ is a $(9, 15)$ NA graph, hence the graph illustrated in Figure 6, with degree sequence $(3^6, 4^3)$. Since G has only two vertices of degree 5, vertex a is adjacent to at most one of x, y, and z in Figure 6. We will assume that it is x and hence $y, z \notin N(a)$. By Lemma 2.4, a must be near edges incident to both y and z (consider $G-z$ and $G-y$, respectively). However, as a has a unique neighbor of degree 2 in $G-a$, it is near only one edge. Therefore, a is near the edge yz. If a is adjacent to v_1, v_2, and v_3 or w_1, w_2, and w_3 then G is H_{11}.

We next verify that this must be the case. Note that there are exactly three vertices in $N(a) \cup \{v_1, v_2, v_3, w_1, w_2, w_3\}$. Let us first examine the intersection with $\{v_2, v_3, w_2, w_3\}$. Lemma 2.4 applied to $G-z$ shows that a has at least one neighbor in each of the pairs $\{v_2, w_3\}$, $\{v_3, w_2\}$, and $\{v_3, w_3\}$. The same lemma with $G-x$ shows that $N(a) \cap \{v_2, v_3, w_2, w_3\}$ is not simply $\{v_3, w_3\}$. We conclude that a is adjacent to w_2 and w_3 or v_2 and v_3, and, by symmetry, we can assume v_2 and v_3. The last neighbor of a must be v_1, as otherwise $G - v_3, w_3$ or $G - v_2, w_2$ will be planar.

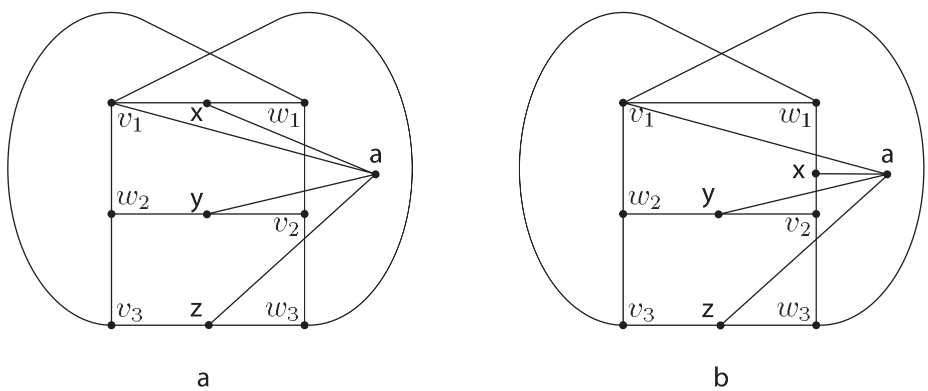

FIGURE 10. Graphs with degree sequence $(3^8, 4^2)$ by adding a degree 4 vertex a to a split $K_{3,3}$.

Let a and b be the degree 5 vertices and suppose neither has a degree 3 neighbor. If a and b are not adjacent, then $(G-a,b)^s$ is a (3^4) graph that is clearly planar. Further, a and b can have at most three common neighbors, as otherwise $(G-a,b)^s$ has fewer than nine edges and is therefore planar. On the other hand, since there are only five degree 4 vertices, a and b must share at least three neighbors. This means $(G-a,b)^s = K_{3,3}$. By Lemma 2.7, $G-b$ must be as in Figure 10a or b. By our assumption, b is adjacent to a, x, y, and z, with one other neighbor from the set $\{w_1, w_2, w_3, v_2, v_3\}$. In the case where $G-b$ looks like Figure 10a we see that $G - v_1, w_1$ is planar. For the case of graph b in the figure, observe that $G - v_1, x$ is

planar. Hence if a and b have no degree 3 neighbors, then G is 2–apex. Therefore G must be H_{11}. □

Lemma 6.4. *Let G be an $(11, 21)$ MMN2A graph with degree sequence $(3^3, 4^7, 5)$. Then G is N'_{11}.*

Proof. Let us begin by assuming that the degree 5 vertex, call it b, is adjacent to some vertex of degree 3. Then $G^* = (G - b)^s$ has degree sequence $(3^6, 4^3)$ and is therefore the P_9 graph of Figure 6. Note that b is not adjacent to x, y, or z, since going from G to G^* did not change their degree. However, observing the graphs we obtain when removing x, y, or z, by Lemma 2.4 we see that b needs a path to all of them that does not utilize any of their neighbors in G^*. This is clearly impossible since there is at most one subdivision from G^* to $G - b$. Hence $\forall v \in N(b)$ we have $\deg(v) = 4$.

Then $G - b$ must have the degree sequence $(3^8, 4^2)$. If the vertices of degree 4 in $G - b$ are not adjacent, then if v is one of those, $(G - b, v)^s$ has eight edges and is therefore planar, which is a contradiction. So choose $a \in V(G - b)$ such that $\deg(a) = 4$. Then if G is N2A, $(G - a, b)^s$ is $K_{3,3}$. When we add a back in, by Lemma 2.7, there are two cases, shown in Figure 10. However, for Figure 10b, we notice that b is not adjacent to v_1 since it can only be adjacent to vertices of degree 3 in $G - b$. This means that it is not near v_1 which is required by Lemma 2.4. So $G - b$ is isomorphic to the graph illustrated in Figure 10a. As above, since b must be near v_1, it must be adjacent to x. Now, $G - v_1, w_1$ will be planar unless $N(b)$ includes either $\{v_2, v_3\}$ or $\{w_2, w_3\}$. We will argue that it must be the latter. Suppose instead $\{x, v_2, v_3\}$ is in $N(b)$ and $\{w_2, w_3\}$ is not. In particular, if $w_2 \notin N(b)$, then $G - v_3, w_3$ is planar, a contradiction. Similarly, if $w_3 \notin N(b)$, $G - v_2, w_2$ gives a contradiction. This shows that it is not possible that $\{w_2, w_3\} \not\subset N(b)$, and so we can assume $\{w_2, w_3\} \subset N(b)$. Now $G - v_2, w_2$ is planar unless b is adjacent to y and $G - v_3, w_3$ shows z is adjacent to b as well, which means G is N'_{11}. □

Lemma 6.5. *Let G be an $(11, 21)$ MMN2A graph with degree sequence $(3^2, 4^9)$. Then G is N_{11}.*

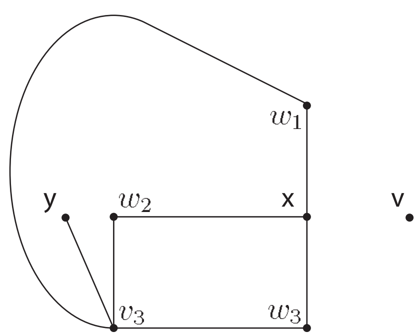

FIGURE 11. Remove v_1 and v_2 from $K_{4,4} - e$.

Proof. First assume that there exists a $v \in V(G)$ such that $\deg(v) = 4$ and the two vertices of degree 3 are neighbors of v. Then $(G - v)^s$ has degree sequence $(3^2, 4^6)$ and is the Petersen family graph $K_{4,4} - e$ illustrated in Figure 8. Thus $G - v$ is a

subdivision of $K_{4,4} - e$. Note that in G, vertex v is adjacent to both x and y. The graph obtained from $K_{4,4} - e$ when we remove v_1 and v_2 is illustrated in Figure 11. Since v is adjacent to both x and y and the graph $G - v, v_1, v_2$ can be obtained from Figure 11 by only two subdivisions (the other neighbors of v), we see that $G - v_1, v_2$ is planar.

FIGURE 12. There are two or four edges between V_3 and V_4.

We can now assume that the two degree 3 vertices of G have no common degree 4 neighbors. Let a be a degree 4 vertex that has a degree 3 neighbor. Then $G^* = (G - a)^s$ has degree sequence $(3^4, 4^5)$. Notice first that if G^* has a degree 4 vertex v that has three or more degree 3 neighbors, then $(G^* - v)^s$ has at most 9 edges and 5 vertices and is planar. We claim that there is a degree 4 vertex in G^*, that has two neighbors of degree 3. For suppose not and let V_3 denote the set of degree 3 vertices of G^* and V_4 those of degree 4. As the degree sums in the two parts are even, there are an even number of edges between V_3 and V_4. If there were six or more, then, by pigeonhole, one of the degree 4 vertices would have two degree 3 neighbors, which is what we are trying to establish. If there were no edges in between, $G^* = K_4 \sqcup K_5$ is apex, a contradiction. So there are two or four edges between V_3 and V_4. (See Figure 12.) In either case, removing a degree 4 vertex that has a degree 3 neighbor will result in a planar graph.

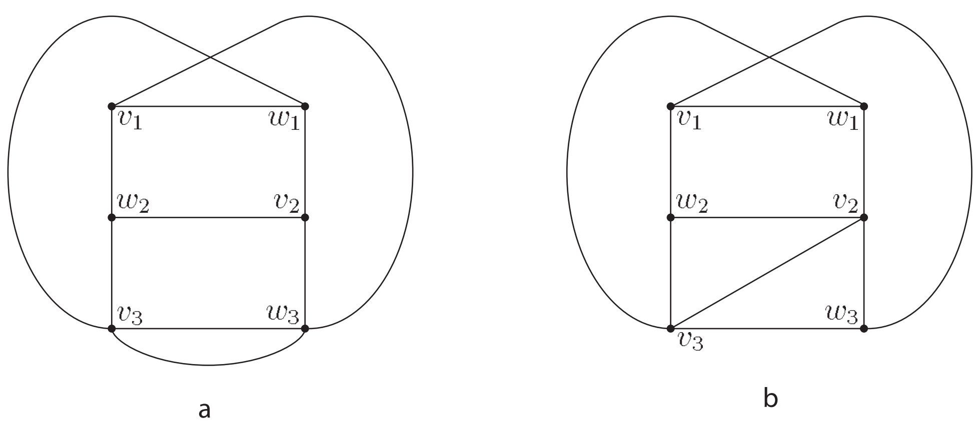

FIGURE 13. Two non-planar $(6, 10)$ graphs.

So, let $b \in V(G^*)$ be a degree 4 vertex with two degree 3 neighbors. Moreover, a and b have a common neighbor, as otherwise b has two degree 3 neighbors in G. Now, $G^* - b$ will be formed by subdividing two edges of a $(6, 10)$ graph G' having

degree sequence $(3^4, 4^2)$. Since our assumption implies that G' is non-planar, G' is one of the two graphs obtained by adding an edge to $K_{3,3}$ (see Figure 13).

Assume that G' is the multigraph $K_{3,3} + e$ shown in Figure 13a. Since G was a simple graph, there is at least one subdivision on one of the paired edges. If a and b are not near the same edge in the paired edges, then removing from G vertices v_3 and w_3 of G' results in a planar graph, since the graph is essentially a subdivision of the 4-cycle $v_1w_1v_2w_2$ along with two more vertices that are not adjacent to one another.

Next, suppose a and b are adjacent to the same edge in the pair, but attach to the edge at two different vertices formed by subdividing that edge twice. By generalizing the argument of Lemma 2.4, we claim that both a and b must have paths to each of the vertices in G' independent of the other vertices of G'. For example, without loss of generality and referencing Figure 13a, if no such path from a to v_2 exists, then $G - b, w_3$ must be planar. Indeed, place a in the region of $G' - w_3$ bounded by the cycle $v_1w_1v_3w_2$. We can argue similarly for b. Recall that $G - a, b$ is obtained from G' by exactly three edge subdivisions. Also, when we add b to G', it is adjacent to two vertices formed by subdivision and two vertices of degree 3. Using Lemma 2.7, we can assume b is near v_1w_1 via a subdivision of that edge and also adjacent to v_2 and w_2 (by the symmetry of G'). Note that since $\Delta(G) = 4$, there is now no way to make paths from a to v_2 and w_2 that avoid the other vertices of G'.

We conclude that a and b attach at the same vertex of one of the paired edges of G'. Then as above, we can assume that b is near the edge v_1w_1 and adjacent to v_2 and w_2. Then those two vertices have degree 4 and are not adjacent to a. As there remains a single subdivision of G', it must be on the edge v_2w_2. So, a is near that edge which forces a to be adjacent to v_1 and w_1. This graph is N_{11}.

Now assume that G' is the simple graph illustrated in Figure 13b. The graph $G' - v_3$, shows us that both a and b are near w_1, w_2, and w_3. Similarly, $G' - w_3$ shows us that they are near v_3 and v_2. Recall that b is adjacent to two of the degree 3 vertices of G' as well as two vertices formed by subdividing edges of G'.

Suppose b is adjacent to v_1 in $G - a$. Then b is adjacent to one of the w_i for $i \in \{1, 2, 3\}$, and by symmetry, we may assume w_1. Since b is also near the other four vertices in G', we may assume b's other neighbors are vertices resulting from subdivisions of the edges w_2v_2 and v_3w_3. Since a and b share at least one neighbor, we may assume (without loss of generality) that a is adjacent to the same vertex formed by subdividing w_3v_3 of G'.

There must be an additional subdivision of G' giving a neighbor of a. Since $\Delta(G) = 4$, the remaining two neighbors of a are drawn from $\{w_2, w_3\}$ and the vertex on v_2w_2 resulting from its subdivision. Suppose a is adjacent to w_2 and w_3. As it must also be near v_2 and w_1, it is also adjacent to a vertex formed by a subdivision of the edge v_2w_1 in G'. However, in this case v_2 has two neighbors of degree 3, a possibility ruled out at the beginning of the proof.

So assume that a shares two neighbors with b, the two vertices formed by subdividing v_2w_2 and v_3w_3, and is adjacent to exactly one of w_2 and w_3, say w_3. Now, a must be near w_1 but if it is adjacent to a vertex formed by the subdivision of v_1w_1 or v_3w_1, we again have the case of a degree 4 vertex with two degree 3 neighbors (v_1 and v_3 respectively). So it must be that a is adjacent to a vertex resulting from subdivision of the edge w_1v_2. In this case, let x denote the common neighbor of a

and b that is also a neighbor of v_3 and w_3. Then $G - x, w_3$ is planar. This shows that b is not adjacent to v_1. A similar argument starting with adding a instead of b shows that a is also not adjacent to v_1, at least in the case where a and b share exactly one neighbor.

So we know that b is not adjacent to v_1 in G'. Then without loss of generality it is adjacent to w_2 and w_3. So, a is adjacent to w_1 or v_1. If a is adjacent to v_1, then a shares two neighbors with b. In other words, the vertices created by subdivisions in going from G' to $G - a, b$ that are neighbors of b are also neighbors of a. Since both a and b are near w_1, suppose they are adjacent to a vertex resulting from subdivision of the edge v_1w_1. Then since a is near w_2, w_3, v_2, and v_3, we may assume a is adjacent to vertices resulting from subdivisions of the edges w_2v_2 and v_3w_3 and that b is adjacent to one of these. However, in either case G has a degree 4 vertex with two degree 3 neighbors (v_3 and v_2 respectively).

So suppose instead that a and b are adjacent to a vertex produced by a subdivision of the edge v_2w_1 (The symmetric case using instead the edge v_3w_1 will be similar.) Since a is near v_3, it must be adjacent to a vertex formed by subdivision of the edge w_2v_3 or w_3v_3 (the other two options will not allow a to be near both w_2 and w_3). Without loss of generality it is w_3v_3. Moreover, this forces b to share this neighbor, as otherwise v_3 will have two degree 3 neighbors in G. The final neighbor of a makes it near w_2 but cannot lie on v_1w_2 or v_3w_2 lest we again have a vertex of degree 4 with two degree 3 neighbors. So a is adjacent to a vertex on the w_2v_2 edge. This is again N_{11}.

Finally, assume that neither a nor b is adjacent to v_1 in G', b is adjacent to w_2 and w_3, and a is adjacent to w_1. The degree 3 vertices in G are then v_1 and the one adjacent to a formed by a subdivision of an edge in G'. Then the two subdivision vertices adjacent to b must also be adjacent to a. Since b is near w_1, assume first that b is adjacent to a subdivision on the edge v_1w_1 in G'. Then the only way to make b near both v_2 and v_3 is by making it adjacent to a vertex formed by subdividing that edge. As a is also adjacent to that vertex, there is no way to make a near both w_2 and w_3. So without loss of generality b (hence a) must be adjacent to a subdivision vertex on the edge v_2w_1 (as the symmetric case where a and b are adjacent to v_3w_1 is similar). Notice now that since a is near both w_2 and w_3 either w_2 or w_3 will share a degree 3 neighbor with a. However, since they are both also neighbors of v_1, G will have a degree 4 vertex with two degree 3 neighbors and cannot be 2–apex. $\square$

Proposition 6.6. *If G is $(11, 21)$ MMN2A, then G is in the Heawood family.*

Proof. Assume that G is an $(11, 21)$ MMN2A graph. As we did in the previous cases, we may assume that the maximal vertex degree of G is 6 or less. Further, if G has more than one vertex of degree 6, then G is not MMN2A, since it must be the case that one of the degree 6 vertices has a degree 3 neighbor and removing such a vertex leaves one with a graph that simplifies to a graph that has no more than 14 edges, hence is not NA by Theorem 1.4. This leaves us with the following degree sequences to consider: $(3^7, 5^3, 6)$, $(3^6, 4^2, 5^2, 6)$, $(3^5, 4^4, 5, 6)$, $(3^4, 4^6, 6)$, $(3^6, 4, 5^4)$,$(3^5, 4^3, 5^3)$, $(3^4, 4^5, 5^2)$, $(3^3, 4^7, 5)$, and $(3^2, 4^9)$.

We can throw out the first three sequences, since it is clear that the degree 6 vertex must have a neighbor of degree 3 and we find ourselves in the same situation as we were in at the beginning of this proof. Five of the remaining six sequences do in fact lead to an MMN2A graph and are treated in the five lemmas above.

This leaves only the degree sequence $(3^6, 4, 5^4)$. Suppose G is a MMN2A graph with this degree sequence. Each degree 5 vertex v has at most one degree 3 neighbor as otherwise $G - v$ simplifies to a graph of at most 14 edges and is not NA by Theorem 1.4. This implies that the vertices of degree 4 and 5 when considered separately, induce a K_5 subgraph, with four of the vertices having other neighbors in G. Choose $a, b \in V(G)$ such that $\deg(a) = \deg(b) = 5$, and consider $G - a, b$. Observe that the induced K_5 subgraph becomes a K_3 subgraph when a and b are removed and only two of its three vertices have neighbors in the rest of $G - a, b$. This means $(G - a, b)^s$ has at most eight edges and is planar, a contradiction. Therefore there is no $(11, 21)$ MMN2A graph G with degree sequence $(3^6, 4, 5^4)$. Together with our five lemmas, this completes the proof. □

7. 10 VERTEX GRAPHS

In this section we prove that a $(10, 21)$ MMN2A graph is in the Heawood family. This is a corollary of the following proposition, originally proved in [BM].

Proposition 7.1. *Let G be a graph with either $|V(G)| \leq 8$ or else $|V(G)| \leq 10$ and $|E(G)| \leq 21$. If G is N2A and a $\mathrm{Y}\nabla$ move takes G to G', then G' is also N2A.*

Proof. Since a graph of 20 or fewer edges is 2–apex [Ma], the only N2A graph with $|G| \leq 7$ is K_7, which has no degree three vertices. So, the proposition is vacuously true for graphs of order seven or less.

Suppose G is N2A with $|G| = 8$. As discussed in [Ma], G must be IK and we refer to the classification of such graphs due independently to [CMOPRW] and [BBFFHL]. There are 23 IK graphs on eight vertices, but only four have a vertex of degree three. In each case, a $\mathrm{Y}\nabla$ move on that vertex results in K_7, which is also N2A.

Again, graphs of size 20 or smaller are 2–apex. So, we can assume $\|G\| = 21$ and $|G| \geq 9$. If G is of order nine and N2A, then, by [Ma, Proposition 1.6], G is a Heawood graph (possibly with the addition of one or two isolated vertices). A $\mathrm{Y}\nabla$ move results in the Heawood graph H_8 or $K_7 \sqcup K_1$, both of which are N2A.

This leaves the case where $|G| = 10$. Assume G is a $(10, 21)$ N2A graph that admits a $\mathrm{Y}\nabla$ move to G'. For a contradiction, suppose G' is 2–apex with vertices a and b so that $G' - a, b$ is planar. Let v_0 be the degree three vertex in G at the center of the $\mathrm{Y}\nabla$ move and v_1, v_2, v_3 the vertices of the resultant triangle in G'. Since G is N2A, it must be that $\{v_1, v_2, v_3\}$ is disjoint from $\{a, b\}$. Fix a planar representation of $G' - a, b$. The triangle $v_1v_2v_3$ divides the plane into two regions. Let H_1 be the induced subgraph on the vertices interior to the triangle and H_2 that of the vertices exterior. Then $|H_1| + |H_2| = 4$. Since G is N2A, there is an obstruction to converting the planar representation of $G' - a, b$ into a planar representation of $G - a, b$. This means that both H_1 and H_2 contain vertices adjacent to each of the triangle vertices $\{v_1, v_2, v_3\}$. In particular, H_1 and H_2 each have at least one vertex.

Suppose $|H_1| = |H_2| = 2$. The graph $G - b, v_1$ is non-planar, but, its subgraph $G - a, b, v_1$ is essentially a subgraph of $G' - a, b$ (with the addition of a degree two vertex v_0 on the edge v_2v_3) and we will use the same planar representation for $G - a, b, v_1$ that we have for $G' - a, b$.

Since $G - b, v_1$ is not planar, there's an obstruction to placing a in the same plane. If we imagine putting a outside of a disk in the plane that covers $G - a, b, v_1$, we

see that their is some vertex w in an H_i that is **hidden** from a. That is, although there's an edge $aw \in E(G)$, there is no a-w path in the plane that avoids $G - b, v_1$. It follows that there's a cycle in $G - b, v_1$ with w interior and a exterior the cycle.

Without loss of generality, the hidden vertex w is in $V(H_1) = \{c_1, d_1\}$, say $w = c_1$. This means we can assume that $c_1v_2d_1v_3$ is a 4–cycle in G, which, in the planar embedding of $G' - a, b$, is arranged with c_1 interior to the cycle $v_2d_1v_3$. However, since $G' - a, b$ is planar, this means c_1 is also hidden from v_1 and c_1v_1 is not an edge of the graph.

A similar argument using $G - b, v_2$ allows us to deduce a 4–cycle $c_2v_1d_2v_3$ using the vertices c_2 and d_2 of H_2 while showing $c_2v_2 \notin E(G)$. However, it follows that $G - b, v_3$ is planar, a contradiction.

So, we can assume $|H_1| = 3$ while H_2 consists of the vertex c_2 with $\{v_1, v_2, v_3\} \subset N(c_2)$. Suppose H_1 also has a vertex, c_1, that is adjacent to all three triangle vertices. As $G - b, v_1$ is non-planar, there's a vertex of H_1, call it d_1, that is hidden from a such that $c_1v_2d_1v_3$ is a cycle in G and $d_1v_1 \notin E(G)$. Similarly, $G - b, v_2$ shows that $c_1v_1e_1v_3$ is in G and e_1v_2 is not, e_1 being the third vertex of H_1. Now, $G - b, v_3$ will be planar unless $d_1e_1 \in E(G)$. However, in that case, contracting d_1e_1 shows that $G' - a, b$ has a $K_{3,3}$ minor and is non-planar, a contradiction.

If H_1 has no vertex c_1 that, on its own, is adjacent to the three triangle vertices, then either H_1 is connected, or else it is not but has an edge c_1d_1 such that $\{v_1, v_2, v_3\} \subset N(c_1) \cup N(d_1)$. But, in this latter case, we can rearrange the planar representation of $G' - a, b$ such that the third vertex of H_1 is exterior to the triangle, returning to the earlier case where $|H_1| = |H_2| = 2$. So we will assume H_1 is connected.

Suppose H_1 is not complete, having only two edges c_1d_1 and d_1e_1. Again $G - b, v_1$ shows that at least two vertices of H_1 are in $N(v_2) \cap N(v_3)$ and there are two cases depending on whether or not $\{c_1, e_1\} \subset N(v_2) \cap N(v_3)$. If both c_1 and e_1 are in the intersection, then we can assume c_1 is hidden from a, meaning $ac_1 \in E(G)$, but $c_1v_1 \notin E(G)$. Actually, since c_1 is interior to the cycle $v_2e_1v_3$, it follows that d_1 is as well and $d_1v_1 \notin E(G)$ either. Then e_1 is the unique vertex of H_1 adjacent to v_1 and $G - b, v_2$ is planar, which is a contradiction.

If c_1 and e_1 are not both in $N(v_2) \cap N(v_3)$, we can assume that c_1 and d_1 are the common vertices with at most one of those adjacent to v_1. If $c_1v_1 \notin E(G)$, then $G - b, v_2$ shows $d_1v_1e_1v_3$ is in G and e_1v_2 is not. But then $G - b, v_3$ is planar, a contradiction. So, we can assume it must be d_1 that's hidden, meaning ad_1 is an edge and d_1v_1 is not. In this case, $G - b, v_2$ must be planar, a contradiction.

Finally, if $H_1 = K_3$, then a similar sequence of arguments shows that, in G', the induced subgraph on $V(H_1) \cup \{v_1, v_2, v_3\}$ is the octahedron graph and that a and b are both adjacent to the three vertices of H_1. By counting edges, we see that, in fact, a and b each have degree three and we have accounted for all edges in G'. Applying the ∇Y move to recover G, we observe that G is 2–apex (for example, $G - c_1, d_1$ is planar for any pair of vertices $c_1, d_1 \in V(H_1)$), a contradiction.

We've shown that assuming G' is 2–apex leads to a contradiction. Thus, the proposition also holds in the case $|G| = 10$, which completes the proof. □

Corollary 7.2. *If G is a $(10, 21)$ MMN2A graph, then G is in the Heawood family.*

Proof. Suppose G is $(10, 21)$ MMN2A. Recall that $\delta(G) \geq 3$ as otherwise a vertex deletion or edge contraction on a small degree vertex gives a proper minor that is also N2A.

In [Ma], we showed that a graph of order nine is MMN2A if and only if it is in the Heawood family. So, if G has a degree three vertex, then apply a Y∇ move at that vertex to get a graph G'. Then, by Proposition 7.1 and the classification of MMN2A graphs of order nine, G' is Heawood, whence G is too. So, we can assume $\delta(G) \geq 4$ which means the degree sequence of G is either $\{4^8, 5^2\}$ or $\{4^9, 6\}$.

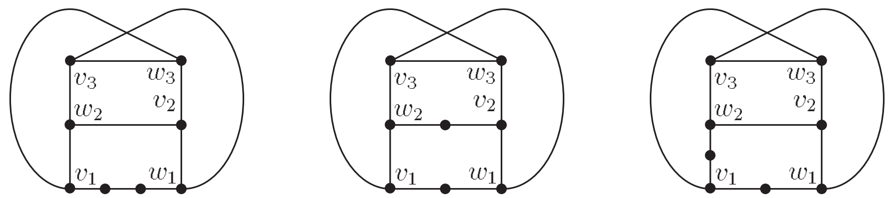

Figure 14. The three non-planar (8,11) graphs of minimal degree at least two.

Suppose there are vertices a and b such that $\|G - a, b\| = 11$. Then at least one of a and b has degree five or six. Since $\delta(G) = 4$, then $\delta(G - a, b) \geq 2$ and $G - a, b$ is one of one of the graphs of Figure 14. In all three cases, both a and b must be adjacent to both v_3 and w_3. For if, for example, a and v_3 are not adjacent, then $G - b, w_3$ would be planar. But, if a and b are adjacent to both, then v_3 and w_3 also have degree five in G, which contradicts the two given degree sequences for G. We conclude there is no choice a and b such that $\|G - a, b\| = 11$.

This means G must have degree sequence $\{4^8, 5^2\}$ with the two vertices of degree five adjacent and $G - a, b$ a $(8, 12)$ graph. There are two cases depending on whether or not a and b have a common neighbor in G. Suppose first that c is adjacent to both a and b. In $G - a, b$ vertex c will have degree two and we can contract an edge on c, to arrive either at a $(7, 11)$ graph or else a multigraph with a doubled edge. Removing the extra edge if needed, let H denote the resulting $(7, 11)$ or $(7, 10)$ graph.

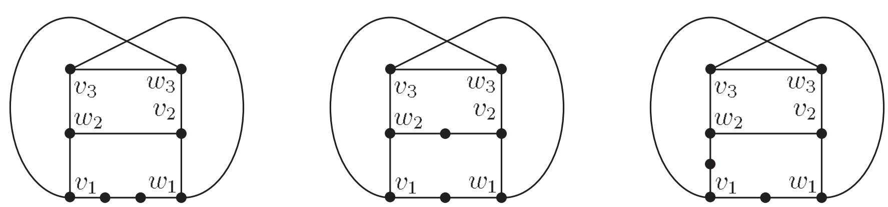

Figure 15. The two non-planar (7,10) graphs of minimal degree at least one.

If H is $(7, 10)$, it is one of the two graphs of Figure 15. In the case of the graph on the left, the doubled edge must be that incident on the degree one vertex as $\delta(G - a, b) \geq 2$. But then the vertex labelled v_1 in the figure will have degree five in $G - a, b$, contradicting our assumption that a and b were the only vertices of degree greater than four. So, we can assume H is the graph to the right in the figure. Up to symmetry, the doubled edge of H is either uv_1, v_1w_2, or v_2w_2. We'll examine the first case; the others are similar. Doubling uv_1 and adding back c leaves v_1 of

degree four in $G - a, b$. Then $G - a, b, v_1$ simplifies to $K_{3,3} - v_1$. Since w_1, w_2, and w_3 all have degree three in $G - a, b$, they each have exactly one of a and b as a neighbor in G. Suppose a is adjacent to w_2. Then $G - a, v_1$ is planar, contradicting G being N2A. For the other two choices of edge doubling, once can again delete a resulting degree four vertex along with a or b to achieve a planar graph. So H being $(7, 10)$ leads to a contradiction.

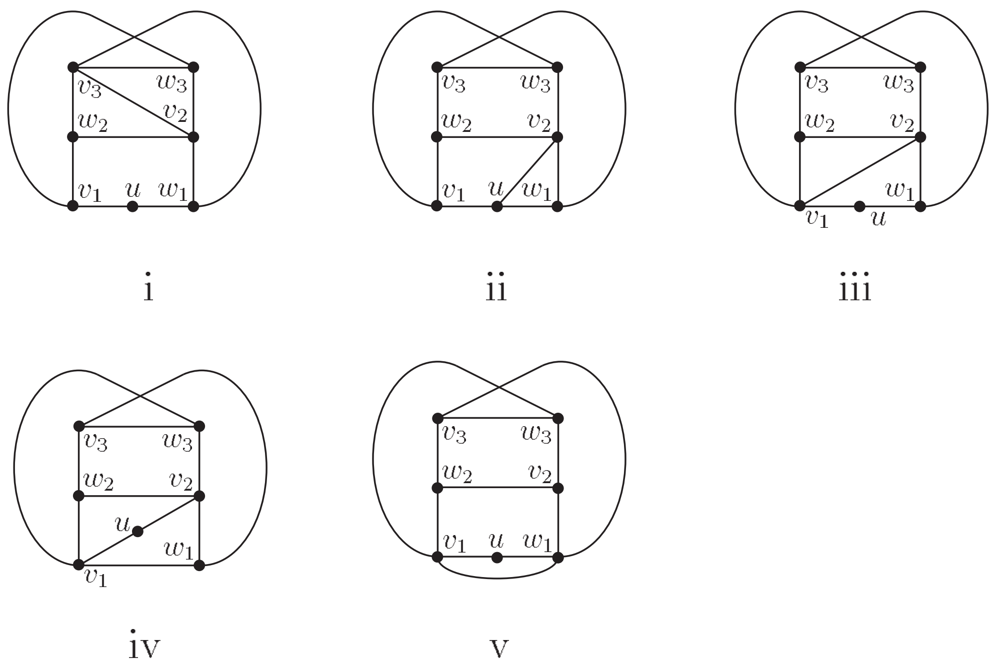

FIGURE 16. The five non-planar (7,11) graphs of minimal degree at least two.

If H is $(7, 11)$, then $\delta(H) = \delta(G - a, b) \geq 2$ and H is one of the five graphs of Figure 16. Here we use a similar approach. Deleting one of the degree four vertices of H, call it x, results in a graph $G - a, b, x$ that simplifies to $K_{3,3} - v_1$. Since each of the degree three vertices of H is adjacent to exactly one of a and b, there will be an appropriate choice from those two, say a, such that $G - a, x$ is planar, which is a contradiction. So, H being $(7, 11)$ is not possible and we conclude that there is no such vertex c that is adjacent to both a and b.

This means that $G - a, b$ is a non-planar cubic graph (i.e., 3-regular) on eight vertices. There are two such graphs, shown in Figure 17. If $G - a, b$ is the graph to the left in Figure 17, note that the vertex labelled v is adjacent to exactly one of a and b, say a. Then $G - a, w$ is planar.

Finally, assume that $G - a, b$ is the graph to the right in Figure 17. Note that each vertex of $G - a, b$ is adjacent to exactly one of a and b in G. If a and b are adjacent to alternate vertices in the 8–cycle (for example if $\{v_1, v_3, v_5, v_7\} \subset N(a)$ and $\{v_2, v_4, v_6, v_8\} \subset N(b)$), we obtain graph 20 of Figure 1, a Heawood graph. If not, then we must have two consecutive vertices, say v_1 and v_2 that share the same

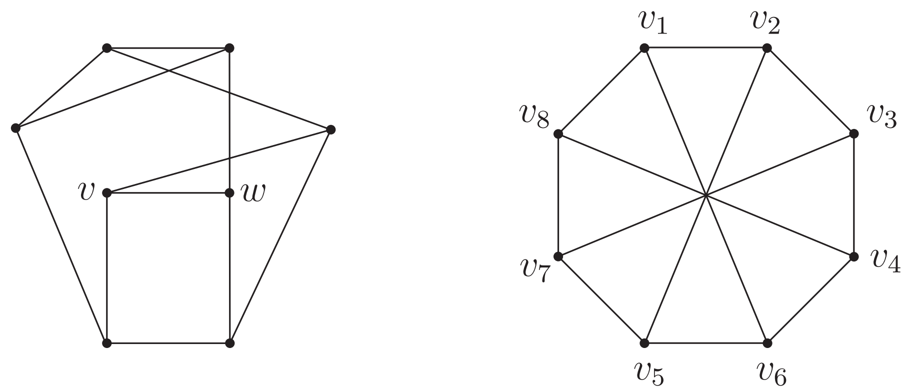

FIGURE 17. The two non-planar cubic graphs of order eight

neighbor in $\{a, b\}$, say a. That is, we can assume $av_1, av_2 \in E(G)$. Then $G - a, v_3$ is planar, contradicting G being N2A.

In summary, if G of order 10 is N2A with $\delta(G) > 3$, it must be graph 20 of the Heawood family. This completes the proof. $\square$

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